HOMOLOGY COMODULES

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Let X be a topological space, $H^*(X)$ the cohomology of X with Z_2 coefficients, A the Steenrod algebra [4] over Z_2, then A acts on $H^*(X)$:

$$\mu: A \otimes H^*(X) \to H^*(X).$$

If we denote by A_* the graded dual of A, $H_*(X)$ the homology of X over Z_2, then

$$\mu_*: H_*(X) \to A_* \otimes H_*(X)$$

makes $H_*(X)$ into a comodule over A_*. Since A_* is much easier to handle than A, it is important to know μ_*. In this note we describe μ_* for $X=BG$ and $X=MG$, where $G=O(n)$, $U(n)$, $Sp(n)$, O, U, Sp.

We remark that Van de Velde [5] determines the coaction for all primes p via an elegant recursive formula involving divided powers. This note describes another algorithm for obtaining the components of the coaction by means of an electronic computer. The algorithm works only for $p=2$.

The paper is organized as follows: the first section describes the initial argument by Van de Velde [5], the second section gives a few immediate results for the coaction in $H_*(RP^\infty)$, the third gives the computer results on $\mu_*(x_{32})$, the fourth points out the implications for the homology of classifying spaces and Thom spaces.

1. **Main formula.** Let A be a graded connected biassociative Hopf algebra (in the application it will be the mod 2 Steenrod algebra) over a field F. Let M be a graded left Hopf algebra over A [2]: that is, M is a left A-module with action

$$\mu: A \otimes M \to M,$$

and multiplication

$$m: M \otimes M \to M,$$

such that the "Cartan formula" is valid—the following diagram is commutative

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$$\begin{tikzcd}
A \otimes M \otimes M \arrow[r, "1 \otimes m"] \arrow[d, "\psi \otimes 1 \otimes 1"'] & A \otimes M \arrow[ddd, "\mu"] \\
A \otimes A \otimes M \otimes M \arrow[d, "1 \otimes \tau \otimes 1"'] & \\
A \otimes M \otimes A \otimes M \arrow[d, "\mu \otimes \mu"'] & \\
M \otimes M \arrow[r, "m"] & M,
\end{tikzcd}$$

where ψ is the coproduct of A, τ the twist map. If

$$d \colon M \to M \otimes M$$

is the coproduct of M, then the following diagrams are commutative

$$\begin{tikzcd}
M \otimes M \arrow[r, "d \otimes d"] \arrow[dd, "m"'] & M \otimes M \otimes M \otimes M \arrow[d, "1 \otimes \tau \otimes 1"] \\
& M \otimes M \otimes M \otimes M \arrow[d, "m \otimes m"] \\
M \arrow[r, "d"] & M \otimes M
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes M \arrow[r, "1 \otimes d"] \arrow[ddd, "\mu"'] & A \otimes M \otimes M \arrow[d, "\psi \otimes 1 \otimes 1"] \\
& A \otimes A \otimes M \otimes M \arrow[d, "1 \otimes \tau \otimes 1"] \\
& A \otimes M \otimes A \otimes M \arrow[d] \\
M \arrow[r, "d"] & M \otimes M
\end{tikzcd}$$

We assume, furthermore, that M and A are finite dimensional over F in each grading.

If we denote by N_* the graded dual of N, then A_* becomes a Hopf algebra with product ψ_*, M_* becomes a Hopf algebra over A_* with coaction μ_*, which satisfies:

(C) $\qquad (1 \otimes m_*) \circ \mu_* = (\psi_* \otimes 1 \otimes 1) \circ (1 \otimes \tau \otimes 1) \circ (\mu_* \otimes \mu_*) \circ m_*.$

Suppose $M=F[u]$, a polynomial algebra over F on one generator u, $\deg u=s\geqq 1$, s even if char $F\neq 2$. Then $d(u)=u\otimes 1+1\otimes u$, and $M_*=\Gamma[\bar{u}]$, a divided polynomial algebra [2] on a class $\bar{u}$ of degree s. If we denote by x_i the class dual to u^i in M_{*si}, then the multiplication in M_* is given by

$$x_ix_j = \binom{i+j}{i} x_{i+j},$$

and the coproduct by

$$m_*(x_n) = \sum_{i+j=n} x_i \otimes x_j.$$

Let us define the elements $\gamma_i^{(n)}$ in A_* by

$$\mu_*(x_n) = \sum_i \gamma_i^{(n)} \otimes x_i.$$

We note that $\deg \gamma_i^{(n)}=(n-i)s$, $\gamma_n^{(n)}=1$, $\gamma_i^{(n)}=0$ if $i<0$ or $i>n$.

Proposition 1 (Van de Velde). *For each n and each pair of integers s, t such that $s+t=i$, we have*

(MF) $$\gamma_i^{(n)} = \sum_a \gamma_s^{(a)}\gamma_t^{(n-a)}.$$

Proof. Apply formula (C) to x_n:

$$(1\otimes m_*)\mu_*(x_n) = \sum_i \gamma_i^{(n)}\Big(\sum_j x_j\otimes x_{i-j}\Big),$$

$$(\psi_*\otimes 1\otimes 1)(1\otimes\tau\otimes 1)(\mu_*\otimes\mu_*)m_*(x_n)$$

$$= (\psi_*\otimes 1\otimes 1)(1\otimes\tau\otimes 1)\sum_a\Big(\sum_s \gamma_s^{(a)}\otimes x_s\Big)\Big(\sum_t \gamma_t^{(n-a)}\otimes x_t\Big)$$

$$= \sum_a\sum_{s,t}\gamma_s^{(a)}\gamma_t^{(n-a)}\otimes x_s\otimes x_t.$$

Comparing coefficients:

$$\gamma_i^{(n)} = \sum_a \gamma_s^{(a)}\gamma_t^{(n-a)},$$

which is the desired formula.

Corollary 2. *If $\gamma_0^{(n)}$ and $\gamma_1^{(n)}$ are known for all n, then $\gamma_i^{(n)}$ can be determined recursively by formula* (MF).

2. **Projective spaces.** Let $M=H^*(RP^\infty;Z_2)=Z_2[u]$, $\deg u=1$; let A be the Steenrod algebra over Z_2. The elements $\alpha_i\in A_{*2^i-1}$ are defined by the equation [4]:

$$\vartheta u = \langle\alpha_i,\vartheta\rangle u^{2^i} \quad \text{for } \vartheta\in A_{2^i-1}.$$

It is also known that for all $a \in A_q$, $au=0$ unless $q=2^i-1$, and $a1=0$ for all deg $a>0$. Let x_i be the dual of u^i. The equations above imply:

$$\begin{aligned} \gamma_0^{(n)} &= 0 \quad \text{for } n > 0, \\ \gamma_1^{(n)} &= 0 \quad \text{if } n \neq 2^i, \\ \gamma_1^{(n)} &= \alpha_i \quad \text{if } n = 2^i. \end{aligned}$$

According to Corollary 2, we can now determine all of the coefficients $\gamma_i^{(n)}$ using formula (MF). Several simplifications are possible:

LEMMA 3. *For all n, r*

$$\begin{aligned} \gamma_{2^r}^{(n)} &= 0 \qquad \textit{if } n \neq 2^{r+i}, \\ &= \alpha_i^{2^r} \quad \textit{if } n = 2^{r+i}, \end{aligned}$$

Proof. The lemma is known for $r=0$. Suppose $r \geqq 1$, let $s=2^{r-1}$, $t=2^{r-1}$ in formula (MF) and notice that all terms cancel out in pairs, except the term $\gamma_{2^{r-1}}^{(k)}\gamma_{2^{r-1}}^{(k)}$ if $n=2k$.

COROLLARY 4 (PERIODICITY). *For all $r \geqq 0$, $i \geqq 0$, and $n<2^{r+1}-2^r$,*

$$\gamma_{i+2^r}^{(n+2^r)} = \gamma_i^{(n)}.$$

Proof. According to formula (MF)

$$\gamma_{i+2^r}^{(n+2^r)} = \sum_a \gamma_i^{(n+2^r-a)}\gamma_{2^r}^{(a)}.$$

However, $\gamma_{2^r}^{(0)}=0$, $\gamma_{2^r}^{(2^r)}=1$, and $\gamma_{2^r}^{(a)}=0$ for $2^r<a<2^{r+1}$; since $n<2^{r+1}-2^r$, $n+2^r<2^{r+1}$, thus $a<2^{r+1}$, and the formula reduces to

$$\gamma_{i+2^r}^{(n+2^r)} = \gamma_i^{(n)}.$$

Corollary 4 asserts that if we plot $\gamma_i^{(n)}$ in a table with respect to n, i, then the resulting table is triangular, has entries 1 along the line $n=i$, and is periodic near the diagonal with period 2^r (the periodicity being a $2^r \times 2^r$ matrix). The periodic blocks B_r for $r \leqq 4$ are given in the first sequence of tables. To simplify the tables, we notice that Corollary 4 implies that

$$B_{r+1} = \begin{pmatrix} B_r & 0 \\ C_r & B_r \end{pmatrix},$$

hence we will only give the matrices C_r.

TABLE I. C_1

	1
1	α_1

TABLE II. C_2

	1	2
3	0	0
4	α_2	α_1^2

TABLE III. C_3

	1	2	3	4
5	0	0	α_1^2	0
6	0	0	$\alpha_1^3+\alpha_2$	0
7	0	0	0	0
8	α_3	α_2^2	$\alpha_2\alpha_1^2$	α_1^4

TABLE IV. C_4

	1	2	3	4	5	6	7	8
9	0	0	α_2^2	0	α_1^4	0	α_1^2	0
10	0	0	$\alpha_2^2\alpha_1+\alpha_3$	0	α_1^5	α_1^4	$\alpha_1^3+\alpha_2$	0
11	0	0	0	0	0	0	α_1^4	0
12	0	0	$\alpha_2^3+\alpha_3\alpha_1^2$	0	$\alpha_2\alpha_1^4+\alpha_3$	$\alpha_1^6+\alpha_2^2$	$\alpha_1^5+\alpha_2\alpha_1^2$	0
13	0	0	0	0	0	0	$\alpha_1^6+\alpha_2^2$	0
14	0	0	0	0	0	0	$\alpha_1^7+\alpha_2\alpha_1^4+\alpha_2^2\alpha_1+\alpha_3$	0
15	0	0	0	0	0	0	0	0
16	α_4	α_3^2	$\alpha_3\alpha_2^2$	α_2^4	$\alpha_3\alpha_1^4$	$\alpha_2^2\alpha_1^4$	$\alpha_2\alpha_1^6+\alpha_2^3+\alpha_3\alpha_1^2$	α_1^8

3. **The coaction on x_{128}.** We recall that $H_*(RP^\infty)$ is, as an algebra, an exterior algebra on the elements x_{2^r}, for $r=0, 1, 2, \ldots$. Since μ_* is a homomorphism of algebras, to determine $\mu_*(x_n)$ it is sufficient to know $\mu_*(x_{2^r})$ for r entering in the dyadic expansion of n. The computation of $\mu_*(x_n)$ for $n \leqq 128$ was carried out on the IBM 7094. The results for $n=32$ are listed below. In the table below each line is headed by j; after the colon $\gamma_j^{(32)}$ follows.

TABLE V. Coaction on x_{32}

0:	0	16:	α_1^{16}.
1:	α_5	17:	α_4.
2:	α_4^2.	18:	α_3^2.
3:	$\alpha_4\alpha_3^2$.	19:	$\alpha_3\alpha_2^2$.
4:	α_3^4.	20:	α_2^4.
5:	$\alpha_4\alpha_2^4$.	21:	$\alpha_3\alpha_1^4$.
6:	$\alpha_3^2\alpha_2^4$.	22:	$\alpha_2^2\alpha_1^4$.
7:	$\alpha_4\alpha_2^2\alpha_1^4+\alpha_3^3\alpha_1^4+\alpha_3\alpha_2^6$.	23:	$\alpha_3\alpha_1^2+\alpha_2^3+\alpha_2\alpha_1^6$.
8:	α_2^8.	24:	α_1^8.
9:	$\alpha_4\alpha_1^8$.	25:	α_3.
10:	$\alpha_3^2\alpha_1^8$.	26:	α_2^2.
11:	$\alpha_4\alpha_2^2+\alpha_3^3+\alpha_3\alpha_2^2\alpha_1^8$.	27:	$\alpha_2\alpha_1^2$.
12:	$\alpha_2^4\alpha_1^8$.	28:	α_1^4.
13:	$\alpha_4\alpha_1^4+\alpha_3\alpha_2^4+\alpha_3\alpha_1^{12}$.	29:	α_2.
14:	$\alpha_3^2\alpha_1^4+\alpha_2^6+\alpha_2^2\alpha_1^{12}$.	30:	α_1^2.
15:	$\alpha_4\alpha_1^2+\alpha_3^2\alpha_2+\alpha_3\alpha_2^2\alpha_1^4+\alpha_2^5\alpha_1^2$	31.	α_1.
	$+\alpha_3\alpha_1^{10}+\alpha_2^3\alpha_1^8+\alpha_2\alpha_1^{14}$.	32:	1.

4. **Classifying spaces and Thom spaces.** Let $O(n)$ be the orthogonal group in dimension n and $O=\lim_{n\to\infty} O(n)$. Then $BO(1)=RP^\infty$, and $O(1)\to O$ induces

$BO(1) \xrightarrow{j} BO$, which is a monomorphism in homology with Z_2 coefficients. Furthermore, $H_*(BO; Z_2)$ is a polynomial algebra [6], [3] on classes $u_i \in H_i(BO; Z_2)$ defined by

$$\langle u_i, w_1^i \rangle = 1, \qquad \langle u_i, m \rangle = 0,$$

where m is any monomial in w_j not equal to w_1^i. Thus $j_*(x_i) = u_i$, hence we know the coaction on u_i.

The homology of $BO(n)$ is a subcoalgebra of the homology of BO:

PROPOSITION 5. *$H_*(BO(n))$ is the vector subspace of $Z_2[x_1, x_2, \ldots]$ with basis all monomials having $\leqq n$ products of generators.*

Proof. Immediate, using a counting argument.

EXAMPLE. $H_*(BO(2))$ is given in low dimensions by linear combinations of

0 1.

1 x_1.

2 x_1^2, x_2.

3 $x_2 x_1, x_3$.

4 $x_2^2, x_3 x_1, x_4$.

Let the Thom space $MO(n)$ be the mapping cone of $BO(n-1) \to BO(n)$.

PROPOSITION 6. *$\tilde{H}_*(MO(n))$ is the vector subspace of $Z_2[x_1, \ldots, x_n, \ldots]$ having as a basis all monomials consisting of precisely n factors.*

Proof. $\tilde{H}_*(MO(n))$ is the quotient of $H_*(BO(n))$ by $H_*(BO(n-1))$. We notice that $\tilde{H}_*(MO(n))$ is actually a direct summand of $H_*(BO(n))$ as an A_*-comodule, since μ_* preserves the number of products of generators occurring in a monomial. This remark together with the proposition proves:

PROPOSITION 7.

$$\tilde{H}_*(BO) \cong \bigoplus_{n=1}^{\infty} \tilde{H}_*(MO(n))$$

as A_*-comodules.

PROPOSITION 8. *The map in homology induced by*

$$MO(m) \wedge MO(n) \to MO(m+n)$$

is the restriction of the product in $H_*(BO)$.

Let MO be the spectrum $\{MO(n)\}$, where the maps $S^1 \wedge MO(n) \to MO(n+1)$ are given by

$$S^1 \wedge MO(n) \to MO(1) \wedge MO(n) \to MO(n+1).$$

In homology $H_*(MO(n); Z_2) \xrightarrow{g_n} H_*(MO(n+1); Z_2)$ is given by

$$g_n(a) = x_1 \cdot a$$

The homology of MO with Z_2 coefficients is defined to be

$$H_i(MO) = \text{inj lim } H_{n+i}(MO(n)).$$

Thus $H_i(MO) \cong H_i(BO)$ as vector spaces, though not as A_*-comodules. We also note that the maps

$$MO(m) \wedge MO(n) \to MO(m+n)$$

make $H_*(MO; Z_2)$ into an algebra over A_*.

PROPOSITION 9.

$$H_*(MO; Z_2) \cong Z_2[a_1, a_2, \ldots],$$

and the coaction of A_* on a_n is obtained by considering $\mu_*(x_{n+1})$ and replacing x_m by a_{m-1}. Furthermore, a_n is represented by the element $x_1^m x_{n+1}$, where m is large.

Proof. Immediate.

We list the coactions of the first few generators:

$$\begin{aligned}
\mu_*(1) &= 1 \otimes 1,\\
\mu_*(a_1) &= \alpha_1 \otimes 1 + 1 \otimes a_1,\\
\mu_*(a_2) &= 1 \otimes a_2,\\
\mu_*(a_3) &= \alpha_2 \otimes 1 + \alpha_1^2 \otimes a_2 + \alpha_1 \otimes a_1 + 1 \otimes a_3,\\
\mu_*(a_4) &= \alpha_1^2 \otimes a_2 + 1 \otimes a_4,\\
\mu_*(a_5) &= (\alpha_1^3 + \alpha_2) \otimes a_2 + \alpha_1 \otimes a_4 + 1 \otimes a_5,\\
\mu_*(a_6) &= 1 \otimes a_6\\
\mu_*(a_7) &= \alpha_3 \otimes 1 + \alpha_2^2 \otimes a_1 + \alpha_2\alpha_1^2 \otimes a_2 + \alpha_1^4 \otimes a_3 + \alpha_2 \otimes a_4 + \alpha_1^2 \otimes a_5\\
&\quad + \alpha_1 \otimes a_6 + 1 \otimes a_7.
\end{aligned}$$

We note that Proposition 9 corrects formula (3.27) in [3] (this was brought to our attention in a letter by J. C. Su; the proof of the main theorem in [3] goes through without change).

The results for $H_*(BO)$ and $H_*(MO)$ immediately go over to results in $H_*(BU)$, $H_*(MU)$, $H_*(BSp)$, $H_*(MSp)$, for the maps

$$H_*(BO) \xrightarrow{f} H_*(BU) \xrightarrow{g} H_*(BSp),$$

$$H_*(MO) \xrightarrow{h} H_*(MU) \xrightarrow{k} H_*(MSp),$$

and the kernels are well known:

PROPOSITION 10.

(1) Ker f = *ideal generated by* x_{2i+1}, *all* i;

(2) $H_*(BU)=Z_2[y_1, y_2, \ldots]$, deg $y_i=2i$, *and* $y_i=f(x_{2i})$;

(3) Ker g = *ideal generated by* y_{2i-1}, *all* i;

$H_*(BSp)=Z_2[z_1, z_2, \ldots]$, deg $z_i=4i$, *and* $z_i=g(y_{2i})$.

(4) *If in* (1), (2), (3) *we replace* B *by* M, f *by* h, g *by* k, x_i *by* a_i, y_i *by* b_i, z_i *by* c_i, *then the statements remain true.*

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